Math 742 Complex Variables Exam 2

The following problems were featured on qualifying exams in complex analysis at the graduate center. Submit your work on any 5.

1. Find the value of

$$\int_C \frac{dz}{1+z^2}$$

(a) When C is the circumference $|z-i|=1$

(b) When C is the circumference $|z|=2$

2. Suppose $f\colon \mathbb{D} \to \mathbb{C}$ is holomorphic and $z_0 \in \mathbb{D}$. Show that f is not injective on a neighborhood of z_0 if and only if $f'(z_0)=0$.

3. Find the number of zeros of $f(z)=z^6-5z^4+3z^2-1$ in $|z|\leq 1$.

4. Show that the equation

$$ze^{a-z}=1, a>1$$

has precisely one root in the disc $|z|\leq 1$. Show this solution must be positive.

5. Find the number of zeros of $f(z)=\frac{1}{3}e^z-z$ in $|z|\leq 1$.

6. Let f be holomorphic in a region G and for $w\in\mathbb{C}$, let $\mu(w)$ be the number of zeros in G of $f(z)-w$, counted according to multiplicity. Suppose $\mu(w_0)\geq n$, where $w_0\in\mathbb{C}$ and n is a positive integer. Show there exists a neighborhood W of w_0 such that every $w\in W-\{w_0\}$ has at least n distinct pre-images under f in G.

7. Let G be a region and suppose that $f\colon\mathbb{C}\to\mathbb{C}$ is holomorphic and $|f|$ obtains its minimum value at an interior point $a\in G$. Show that either $f(a)=0$ or f is a constant function.

8. Prove there is no holomorphic $f\colon\mathbb{C}\backslash\{0\}\to\mathbb{C}$ such that

$$|f(z)|\geq\frac{1}{\sqrt{|z|}}$$

for all $z\neq 0$.

9. Suppose f is holomorphic in an open set containing the closed unit disk, except for a simple pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disk, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$$

10. Give an example of a closed rectifiable curve γ in $\mathbb{C}$ such that for any integer k there is a point $a \notin \gamma$ with winding number $\eta(\gamma; a) = k$.

11. Suppose f and g are holomorphic in a region Ω containing the closure of the disc $\mathbb{D}_R = \{z: |z| < R\}$ (i.e. $\overline{\mathbb{D}}_R \in \Omega$) and $|f(z)| = |g(z)|$ on $C_R = \partial\mathbb{D}_R$. Show that if neither f nor g vanishes in $\mathbb{D}_R$, there is a constant λ, with $|\lambda| = 1$, such that $f = \lambda g$.